

IMSc/2016/08/03

A Class of LQC–inspired Models for Homogeneous, Anisotropic Cosmology in Higher Dimensional Early Universe

S. Kalyana Rama

Institute of Mathematical Sciences, HBNI, C. I. T. Campus,
Tharamani, CHENNAI 600 113, India.

email: krama@imsc.res.in

ABSTRACT

The dynamics of a $(3 + 1)$ dimensional homogeneous anisotropic universe is modified by Loop Quantum Cosmology and, consequently, it has generically a big bounce in the past instead of a big-bang singularity. This modified dynamics can be well described by effective equations of motion. We generalise these effective equations of motion empirically to $(d + 1)$ dimensions. The generalised equations involve two functions and may be considered as a class of LQC – inspired models for $(d + 1)$ dimensional early universe cosmology. As a special case, one can now obtain a universe which has neither a big bang singularity nor a big bounce but approaches asymptotically a ‘Hagedorn like’ phase in the past where its density and volume remain constant. In a few special cases, we also obtain explicit solutions.

1. Introduction

Assume that the universe is $(d + 1)$ dimensional, it is homogeneous and anisotropic, the d -dimensional space is toroidal, and that the universe is dominated by matter fields, characterised by density and anisotropic pressures along the d spatial directions. In Einstein's theory, such a universe generically has a big-bang singularity in the past where its physical volume vanishes, its density diverges, and the curvature invariants diverge. In $(3 + 1)$ dimensions, the early universe dynamics is modified by Loop Quantum Cosmology (LQC) [1, 2, 3] which is derived within Ashtekar's Loop Quantum Gravity (LQG) framework [4]. See the review [5] which also contains an exhaustive list of references. With the dynamics thus modified by quantum effects, the physical volume of the universe does not vanish but reaches a non zero minimum, and its density does not diverge but reaches a finite maximum. There is no singularity in the past. Instead, as one goes back in time starting with a large universe, its volume decreases, reaches a minimum, then bounces back and increases again. Correspondingly, the density increases, reaches a finite maximum, and decreases again. Thus, in LQC, the $(3 + 1)$ dimensional universe generically has a big bounce in the past, instead of a big-bang singularity.¹ This dynamics, modified by the quantum effects, can be well described by effective equations of motion which reduce to the standard Einstein's equations in the 'classical limit'. See [3, 5] for a review.

In this paper, we consider early universe cosmology in $(d + 1)$ dimensional spacetime where $d \geq 3$. We generalise the effective equations in LQC. These generalised effective equations of motion may now describe the modified dynamics of a $(d + 1)$ dimensional homogeneous anisotropic universe. The generalisation we present involves two functions f and ϕ^i , or equivalently $\bar{\mu}^i$, see equations (30) and (31) below. The generalisation is natural and straightforward but empirical : The functions involved are not obtained from an underlying theory but are presented as models. The resulting generalised

¹ In a different quantisation scheme [6, 7], the isotropic evolution of a $(3 + 1)$ dimensional universe with a massless scalar field has no bounce, but undergoes a regular non singular extension past the classical singularity: the volume of the universe vanishes at one time in the past but its density remains bounded all through the evolution. However, it is not known whether such an evolution is generic or whether it is possible in other cases where, for example, matter content is different or evolution is anisotropic.

equations, involving two functions, may therefore be considered as a class of LQC – inspired models for $(d + 1)$ dimensional early universe cosmology.²

The matter sector, in both LQC and in the models presented here, may include various types of scalar fields and other matter fields. But it is assumed to couple to the metric fields but not to the curvatures. This assumption also leads to the standard conservation equation (6) below.

Special cases of the functions in the present models lead to Einstein’s equations in $(d + 1)$ dimensions given, for example, in [12] and to the effective LQC equations in $(3 + 1)$ dimensions given, for example, in [5, 13, 14]. In the former case, there is a big bang singularity. In the later case, there is a big bounce. One can also choose the functions such that the resulting universe has neither a big bang singularity nor a big bounce but, as one goes back in time starting with a large volume, the volume approaches a non zero minimum asymptotically and the density approaches a finite maximum asymptotically. This is similar to what is expected in string/M theory where, as one goes back in time, the ten/eleven dimensional early universe is believed to enter and remain in a ‘Hagedorn phase’ where its temperature $= \mathcal{O}(l_s^{-1})$ and its density $= \mathcal{O}(l_s^{-(d+1)})$, l_s being the string length scale [15] – [22], [12].

The effective equations of motion can be solved numerically for any choice of the functions. The general features of these equations, such as when the density will remain finite or will diverge, can also be seen easily. In general, however, it is not possible to obtain explicit analytical solutions to the equations of motion. They can be obtained in some special cases. In this paper, we will consider a few such cases and present explicit solutions.

This paper is organised as follows : In Section 2, we present the Einstein’s equations for a $(d + 1)$ dimensional homogeneous anisotropic universe. In Section 3, we present a class of LQC – inspired models. We give the generalised effective equations of motion and also show how Einstein’s and effective LQC equations follow as special cases. In Section 4, we present a few explicit solutions. In Section 5, we conclude by mentioning a few issues for further studies.

² Arnab Priya Saha has pointed out to us that there exists a $(d + 1)$ dimensional LQG formulation, given in [8, 9, 10]. Our preliminary analysis [11] suggests that one can derive the LQC analogs of the effective equations in $(d + 1)$ dimensions. In the present empirical framework, this corresponds to a particular choice of the two functions.

2. Einstein's equations

Let the spacetime be $D = d + 1$ dimensional and let x^i , $i = 1, 2, \dots, d$, denote the spatial coordinates. We take the d -dimensional space to be toroidal and let L_i denote the coordinate length of the i^{th} direction. Consider the homogeneous and anisotropic case where the line element ds is given by

$$ds^2 = -dt^2 + \sum_i a_i^2 (dx^i)^2 \quad (1)$$

and the scale factors a_i are functions of t only.³ Let the energy momentum tensor T^A_B , with $A, B = (0, i)$, be diagonal and be given by

$$T^0_0 = -\rho \quad , \quad T^i_i = \hat{p}_i \quad (2)$$

where the density ρ is assumed to be positive and the pressures \hat{p}_i in the i^{th} direction are to be given by the equations of state. Einstein's equations are given, in the standard notation with $\kappa^2 = 8\pi G_D$, by

$$R_{AB} - \frac{1}{2} g_{AB} R = \kappa^2 T_{AB} \quad , \quad \sum_A \nabla_A T^A_B = 0 \quad . \quad (3)$$

Following the notations of our earlier work [12], we define

$$G_{ij} = 1 - \delta_{ij} \quad , \quad G^{ij} = \frac{1}{d-1} - \delta^{ij} \quad \longleftrightarrow \quad \sum_j G^{ij} G_{jk} = \delta^i_k$$

and

$$\Lambda = \sum_i \lambda^i \quad , \quad \lambda^i_t = \frac{(a_i)_t}{a_i} \quad , \quad r^i = \sum_j G^{ij} (\rho - \hat{p}_j) = \hat{p}_i + \frac{\rho - \sum_j \hat{p}_j}{d-1}$$

where the t -subscripts denote time derivatives. Then, after a straightforward algebra, Einstein's equations (3) give⁴

$$\sum_{i,j} G_{ij} \lambda^i_t \lambda^j_t = \Lambda_t^2 - \sum_i (\lambda^i_t)^2 = 2\kappa^2 \rho \quad (4)$$

³In the following, the convention of summing over repeated indices is not always applicable and, hence, will not be followed. We will write explicitly the indices to be summed over.

⁴Note that, written in terms of an average Hubble rate $\frac{a_t}{a} = \frac{\Lambda_t}{d}$ and a shear term

$$\sigma_{shear}^2 = \frac{1}{d(d-1)} \sum_{i,j} (\lambda^i_t - \lambda^j_t)^2 = \frac{2}{d-1} \left(\sum_i (\lambda^i_t)^2 - \frac{\Lambda_t^2}{d} \right) \quad ,$$

$$\lambda_{tt}^i + \Lambda_t \lambda_t^i = \kappa^2 r^i \quad (5)$$

$$\rho_t + \sum_i (\rho + \hat{p}_i) \lambda_t^i = 0 \quad (6)$$

Now, define a variable $\tau(t)$ by

$$dt = e^\Lambda d\tau \quad (7)$$

Then, for any function $X(t)$, equivalently $X(\tau(t))$, we have

$$X_\tau = e^\Lambda X_t \quad , \quad X_{\tau\tau} = e^{2\Lambda} (X_{tt} + \Lambda_t X_t)$$

where the τ -subscripts denote τ -derivatives. In terms of τ , equations (4), (5), and (6) become

$$\sum_{i,j} G_{ij} \lambda_\tau^i \lambda_\tau^j = 2\kappa^2 \tilde{\rho} \quad (8)$$

$$\lambda_{\tau\tau}^i = \kappa^2 \tilde{r}^i \quad (9)$$

$$\tilde{\rho}_\tau = \sum_j (\tilde{\rho} - \tilde{p}_j) \lambda_\tau^j \quad (10)$$

where

$$\tilde{\rho} = e^{2\Lambda} \rho \quad , \quad \tilde{p}_i = e^{2\Lambda} \hat{p}_i \quad , \quad \tilde{r}^i = e^{2\Lambda} r^i = \sum_j G^{ij} (\tilde{\rho} - \tilde{p}_j) \quad (11)$$

Consider the case where the equations of state are given by

$$\hat{p}_i = w_i \rho \quad \longleftrightarrow \quad \rho - \hat{p}_i = u_i \rho \quad (12)$$

equation (4) becomes

$$\Lambda_t^2 - \sum_i (\lambda_t^i)^2 = d(d-1) \left(\frac{a_t}{a}\right)^2 - \left(\frac{d-1}{2}\right) \sigma_{shear}^2 = 2\kappa^2 \rho \quad .$$

where $w_i = 1 - u_i$ are constants. Then

$$\tilde{r}^i = u^i \tilde{\rho} \quad , \quad u^i = \sum_j G^{ij} u_j \quad .$$

Now, in this case, $\lambda^i(\tau)$ and $t(\tau)$ can be obtained explicitly. For this purpose, let

$$l = \sum_i u_i \lambda^i \quad (13)$$

and let the initial values at an initial time t_0 be given by

$$\left(\lambda^i, \lambda_t^i ; \Lambda, l, l_t ; \rho ; \tau \right)_{t=t_0} = \left(\lambda_0^i, k^i ; \Lambda_0, l_0, K ; \rho_0 ; \tau_0 \right) \quad (14)$$

where, as follows from the definitions of Λ and l ,

$$\Lambda_0 = \sum_i \lambda_0^i \quad , \quad l_0 = \sum_i u_i \lambda_0^i \quad , \quad K = \sum_i u_i k^i$$

and k^i and ρ_0 must obey equation (4) and, hence, the relation

$$\sum_{i,j} G_{ij} k^i k^j = 2\kappa^2 \rho_0 \quad .$$

Upon integration, equation (10) gives

$$\tilde{\rho} = \tilde{\rho}_0 e^{l-l_0} \quad , \quad \tilde{\rho}_0 = e^{2\Lambda_0} \rho_0 \quad . \quad (15)$$

Equations (9) and (13) then give an equation for l :

$$l_{\tau\tau} = \kappa^2 U \tilde{\rho}_0 e^{l-l_0} \quad , \quad U = \sum_i u_i u^i = \sum_{i,j} G^{ij} u_i u_j \quad (16)$$

which, together with the initial values l_0 and K , determines $l(\tau)$. Equations (9) and (14) now give

$$\lambda^i - \lambda_0^i = \frac{u^i}{U} (l - l_0) + e^{\Lambda_0} q^i (\tau - \tau_0) \quad (17)$$

$$t - t_0 = \int_{\tau_0}^{\tau} d\tau e^{\Lambda} \quad , \quad \Lambda = \sum_i \lambda^i \quad (18)$$

where

$$q^i = k^i - \frac{u^i}{U} K \quad \longrightarrow \quad \sum_i u_i q^i = 0 \quad .$$

For stiff matter, the equations of state is given by $\hat{p}^i = \rho$. Hence $u_i = u^i = 0$ and $r^i = 0$. Equations (9), (10), and (14) then give

$$\lambda^i - \lambda_0^i = e^{\Lambda_0} k^i (\tau - \tau_0) \quad , \quad \tilde{\rho} = \tilde{\rho}_0 = e^{2\Lambda_0} \rho_0 \quad (19)$$

which can also be obtained from equations (15) and (17) by setting $u_i = u^i = 0$, thus $l - l_0 = K = 0$. Thus, when the equations of state are linear as in (12), $\lambda^i(\tau)$ and $t(\tau)$ are given explicitly by equations (17), or (19), and (18).

3. A class of LQC-inspired models

We first outline briefly the relevant steps involved in obtaining the effective LQC equations for a $(3 + 1)$ dimensional homogeneous anisotropic universe. We will then present our generalisations.

Effective LQC equations

Briefly, the effective equations of motion in LQC may be obtained as follows. For a detailed derivation and for a complete description of various terms and concepts mentioned below, see the review [5]. Let the three dimensional space be toroidal, the line element ds be given by

$$ds^2 = -dt^2 + a_1^2 (dx^1)^2 + a_2^2 (dx^2)^2 + a_3^2 (dx^3)^2 \quad (20)$$

where a_i are functions of t only, and let L_i and $a_i L_i$ be the coordinate and the physical lengths of the i^{th} direction. In the LQG formalism, the canonical pairs of phase space variables consist of an $SU(2)$ connection $A_a^i = \Gamma_a^i + \gamma K_a^i$ and a triad E_i^a of density weight one. Here Γ_a^i is the spin connection defined by the triad e_i^a , K_a^i is related to the extrinsic curvature, and $\gamma > 0$ and ≈ 0.2375 is the Barbero – Immirzi parameter of LQG, its numerical value

being suggested by the black hole entropy calculations. For the anisotropic universe whose line element ds is given by equation (20), one has

$$A_a^i \propto c_i \quad , \quad E_i^a \propto p_i \quad .$$

The full expressions for A_a^i and E_i^a contain various fiducial triads, cotriads, and other elements. They are given in [5, 13] but are not relevant for our purposes here and, hence, not shown. The variables p_i are related to the scale factors a_i and the lengths L_i by

$$p_1 = a_2 a_3 L_2 L_3 \quad , \quad p_2 = a_1 a_3 L_1 L_3 \quad , \quad p_3 = a_1 a_2 L_1 L_2$$

where, with no loss of generality for our purposes here, we have assumed that the physical and the fiducial triads have same orientations and hence p_i are all positive. The variables c_i will turn out to be related to $(a_i)_t$, see [13] and also equation (48) below. The non vanishing Poisson brackets among c_i and p_j are given by

$$\{c_i, p_j\} = \gamma \kappa^2 \delta_{ij}$$

where $\kappa^2 = 8\pi G_4$. The effective equations of motion are given by the ‘Hamiltonian constraint’ $\mathcal{C}_H = 0$ and by the Poisson brackets of p_i and c_i with \mathcal{C}_H which give the time evolutions of c_i and p_i : namely, by

$$\mathcal{C}_H = 0 \quad , \quad (p_i)_t = \{p_i, \mathcal{C}_H\} \quad , \quad (c_i)_t = \{c_i, \mathcal{C}_H\} \quad .$$

Note that it is to be expected that there exists an appropriate ‘classical’ \mathcal{C}_H the Poisson brackets with which lead to the classical dynamics. Non trivially, and as reviewed in detail in [5], there also exists an effective, ‘quantum’ modified, \mathcal{C}_H the Poisson brackets with which lead to the equations of motion which describe the quantum dynamics very well. The effective \mathcal{C}_H reduces to the classical one in a suitable limit.

The expressions for the classical and the effective \mathcal{C}_H are given, for example, in [5, 13, 14]. They are of the form

$$\mathcal{C}_H = H_{grav}(p_i, c_i) + H_{mat}(p_i; \{\phi_{mat}\}, \{\pi_{mat}\}) \quad .$$

The Hamiltonian H_{mat} for a massless scalar field is considered in [5, 13] and that for a general isotropic matter is considered in [14]⁵ where the isotropic

⁵The Hamiltonians \mathcal{H} and the time variable τ considered in these references are related to ours by $\mathcal{H}|_{theirs} = N H|_{ours}$ and $dt|_{ours} = N d\tau|_{theirs}$. The lapse function $N = \sqrt{p_1 p_2 p_3}$ for harmonic time τ .

pressure \hat{P} is given by

$$\hat{P} = - \frac{\partial H_{mat}}{\partial \sqrt{p_1 p_2 p_3}} . \quad (21)$$

The classical H_{grav} , from which Einstein's equations follow, is given by

$$H_{grav} = - \frac{c_1 p_1 c_2 p_2 + c_2 p_2 c_3 p_3 + c_3 p_3 c_1 p_1}{\gamma^2 \kappa^2 \sqrt{p_1 p_2 p_3}} .$$

The effective H_{grav} , from which the LQC dynamics follow, is given in the so-called $\bar{\mu}$ scheme by ⁶

$$H_{grav} = \frac{-1}{\gamma^2 \kappa^2 \sqrt{p_1 p_2 p_3}} \left(\frac{\sin(\bar{\mu}^1 c_1)}{\bar{\mu}^1} \frac{\sin(\bar{\mu}^2 c_2)}{\bar{\mu}^2} p_1 p_2 + \text{cyclic terms} \right)$$

where

$$\bar{\mu}^1 = \lambda_{qm} \sqrt{\frac{p_1}{p_2 p_3}} , \quad \bar{\mu}^2 = \lambda_{qm} \sqrt{\frac{p_2}{p_1 p_3}} , \quad \bar{\mu}^3 = \lambda_{qm} \sqrt{\frac{p_3}{p_1 p_2}}$$

and $\lambda_{qm}^2 = \sqrt{\frac{3}{4}} \gamma \kappa^2$ is the quantum of area. Note that the effective H_{grav} reduces to the classical H_{grav} in the limit $\bar{\mu}^i c_i \ll 1$ for all i .

Generalisations

The effective LQC equations for a $(3 + 1)$ dimensional homogeneous anisotropic universe, given in the previous subsection, can be generalised straightforwardly to $(d + 1)$ dimensions. We now present these generalised equations. Our generalisations are empirical and consist of three simple, straightforward, and natural steps:

- In the LQC expressions given earlier, now let $i = 1, 2, \dots, d$. The anisotropic pressures \hat{p}_i will be proportional to the change in energy per unit physical length in the i^{th} direction. See equation (28) below.
- Introduce a function that generalises the trigonometric functions of the LQC.

⁶For an example of the effective H_{grav} in the so-called μ_0 scheme, see [23].

- Introduce another function that may cover both the $\bar{\mu}$ and the μ_0 schemes.

Our preliminary analysis [11] suggests that, starting from the $(d+1)$ dimensional LQG formulation given in [8, 9, 10], it is possible to derive the LQC analogs of the effective equations in $(d+1)$ dimensions and, along the way, the classical equations also. By introducing two functions, our empirical generalisations here go further beyond what can be derived within LQC framework. However, it is not clear to us if they can be obtained from any underlying theory. In this paper, we simply propose that the resulting equations may be considered as LQC-inspired models for a $(d+1)$ dimensional homogeneous anisotropic universe.

We now proceed with the generalisations. Let the d -dimensional space be toroidal. The canonical pairs of phase space variables are assumed to be given by c_i and p_i where $i = 1, 2, \dots, d$ now. The variables p_i , now generalised to $(d+1)$ dimensions, are given by ⁷

$$p_i = \frac{V}{a_i L_i} \ , \quad V = \prod_j a_j L_j \quad \implies \quad V = \left(\prod_i p_i \right)^{\frac{1}{d-1}} \quad (22)$$

where L_i and $a_i L_i$ are the coordinate and the physical lengths of the i^{th} direction, and V is the physical volume. Thus, p_i is the $(d-1)$ -dimensional physical ‘area’ transverse to the i^{th} direction. Also, define λ^i and l_i by

$$a_i L_i = e^{\lambda^i} \ , \quad V = e^\Lambda \ , \quad p_i = e^{l_i} \\ \longrightarrow \quad l_i = \Lambda - \lambda^i = \sum_j G_{ij} \lambda^j \ , \quad \lambda^i = \sum_j G^{ij} l_j \ . \quad (23)$$

The variables c_i will turn out to be related to $(a_i)_t$, see equation (48) below. We assume that the non vanishing Poisson brackets among c_i and p_j are given by

$$\{c_i, p_j\} = \gamma \kappa^2 \delta_{ij} \quad (24)$$

⁷In the definition of p_i , we have set the orientation related factors $\epsilon_i = +1$ for all i and have assumed that the p_i s are all positive. This suffices for our purposes here.

where $\kappa^2 = 8\pi G_D$ and γ is a constant parameter.⁸ The equations of motion are given by the ‘Hamiltonian constraint’ $\mathcal{C}_H = 0$, and by the Poisson brackets of p_i and c_i with \mathcal{C}_H which give the evolution of c_i and p_i : namely, by

$$\mathcal{C}_H = 0 \quad , \quad (p_i)_t = \{p_i, \mathcal{C}_H\} \quad , \quad (c_i)_t = \{c_i, \mathcal{C}_H\} \quad . \quad (25)$$

It then follows from equation (24) that

$$(p_i)_t = -\gamma\kappa^2 \frac{\partial \mathcal{C}_H}{\partial c_i} \quad , \quad (c_i)_t = \gamma\kappa^2 \frac{\partial \mathcal{C}_H}{\partial p_i} \quad . \quad (26)$$

Consider \mathcal{C}_H . The expressions for \mathcal{C}_H are of the form

$$\mathcal{C}_H = H_{grav}(p_i, c_i) + H_{mat}(p_i; \{\phi_{mat}\}, \{\pi_{mat}\}) \quad . \quad (27)$$

First consider H_{mat} in equation (27). It denotes a generalised matter Hamiltonian which may now include various types of scalar fields and other matter fields, all symbolically denoted as $\{\phi_{mat}\}$ and their conjugate momenta $\{\pi_{mat}\}$. We have assumed that H_{mat} depends only on p_i and is independent of c_i . Since c_i will turn out to be related to $(a_i)_t$, this assumption is equivalent to assuming that matter fields couple to the metric fields but not to the curvatures. This assumption also leads to the conservation equation (6) irrespective of what H_{grav} is: Given H_{mat} , define the density ρ and the pressure \hat{p}_i in the i^{th} direction by

$$\rho = \frac{H_{mat}}{V} \quad , \quad \hat{p}_i = -\frac{a_i L_i}{V} \frac{\partial H_{mat}}{\partial (a_i L_i)} = -\frac{1}{V} \frac{\partial H_{mat}}{\partial \lambda^i} \quad . \quad (28)$$

The pressure \hat{p}_i is thus, as to be physically expected, proportional to the change in energy per unit physical length in the i^{th} direction.⁹ Differentiating

⁸ In the present empirical framework, the parameter γ can be absorbed into the definition of c_i . But we retain it explicitly for the sake of comparisons and also because, in the $(d+1)$ dimensional LQG theories [8, 9, 10], the parameter γ (called β in these works) does enter in the definition of appropriate connection variable. Moreover, in these theories, γ characterises the quantum of the $(d-1)$ -dimensional ‘area’ which is given by $\lambda_{qm}^{d-1} = \mathcal{O}(1) \gamma \kappa^2$. Such a quantisation of area is used in [24] to explain the entropy of $(d+1)$ dimensional black holes.

⁹ If H_{mat} depends on p_i only through volume then its dependence on λ^i is only through $\Lambda = \sum_i \lambda^i$. The pressure $\hat{p}_i = -\frac{1}{V} \frac{\partial H_{mat}}{\partial \Lambda} = -\frac{\partial H_{mat}}{\partial V}$ is then same for all i and is isotropic, cf. equation (21).

ρ , noting that H_{mat} is independent of c_i and depends only on p_i (equivalently λ^i), using $V = e^\Lambda$ and the definition of \hat{p}_i , lead straightforwardly to the conservation equation (6) :

$$\rho_t = \left(\frac{H_{mat}}{V} \right)_t = - \sum_i (\rho + \hat{p}_i) \lambda_t^i .$$

Next consider H_{grav} in equation (27). We now present a straightforward but an empirical generalisation of those expressions for H_{grav} given in [5, 13, 14, 23], now involving two functions. The resulting equations of motion will describe a general class of $d + 1$ dimensional homogeneous anisotropic universe. Let H_{grav} be given by

$$H_{grav} = - \frac{V \mathcal{G}}{\gamma^2 \lambda_{qm}^2 \kappa^2} , \quad \mathcal{G} = \frac{1}{2} \sum_{ij} G_{ij} \psi^i \psi^j = \sum_{ij (i < j)} \psi^i \psi^j \quad (29)$$

where $V = (\prod_i p_i)^{\frac{1}{d-1}}$, λ_{qm} is a length parameter which may be similar to that mentioned in footnote 8, and the fields ψ^i are dimensionless and are to be obtained from an underlying theory, or to be modelled otherwise. In our model, we assume that ψ^i , $i = 1, 2, \dots, d$, are given by

$$\psi^i = \phi^i(p_j) f^i , \quad f^i = f(m^i) , \quad m^i = \bar{\mu}^i(p_j) c_i \quad (30)$$

where the arguments of the functions ϕ^i , f^i , and $\bar{\mu}^i$ are as indicated, and

$$\phi^i \bar{\mu}^i = \frac{\lambda_{qm} p_i}{V} , \quad f(x) \rightarrow x \text{ as } x \rightarrow 0 . \quad (31)$$

The conditions (31) on $(\phi^i, \bar{\mu}^i, f)$ are imposed so that, in the ‘classical limit’ $c_i \rightarrow 0$, one obtains $m^i \rightarrow 0$, $f^i \rightarrow m^i$, and

$$\psi^i = \frac{\lambda_{qm} p_i c_i}{V} , \quad H_{grav} = - \frac{1}{\gamma^2 \kappa^2 V} \sum_{ij (i < j)} (p_i c_i) (p_j c_j) \quad (32)$$

which, as will be seen below, leads to the Einstein’s equations (4) and (5) for a $(d + 1)$ dimensional homogeneous anisotropic universe. Thus, the two functions involved in our generalisation are f and ϕ^i , or equivalently $\bar{\mu}^i$. For $d = 3$, and $f(x) = \sin x$, the effective H_{grav} of Loop Quantum Cosmology given in [5, 13, 14] follows upon setting $\phi^i = 1$, and that given in [23] follows

upon setting $\bar{\mu}^i = \epsilon$ where ϵ is a constant, referred to as the discreteness parameter.

General Equations of motion

Now, from equation (29) for H_{grav} , it follows that

$$\frac{\partial H_{grav}}{\partial c_i} = - \frac{V}{\gamma^2 \lambda_{qm}^2 \kappa^2} \frac{\partial \mathcal{G}}{\partial c_i} \quad (33)$$

$$- \frac{p_i}{V} \frac{\partial H_{grav}}{\partial p_i} = \frac{1}{\gamma^2 \lambda_{qm}^2 \kappa^2} \left(\frac{\mathcal{G}}{d-1} + p_i \frac{\partial \mathcal{G}}{\partial p_i} \right) . \quad (34)$$

From H_{mat} being independent of c_i , from $\lambda^i = \sum_j G^{ij} (\ln p_j)$, and from equation (28), it follows that

$$\frac{\partial H_{mat}}{\partial c_i} = 0 \quad , \quad - \frac{p_i}{V} \frac{\partial H_{mat}}{\partial p_i} = \sum_j G^{ij} \hat{p}_j . \quad (35)$$

Then the equations of motion following from equations (25) are given by

$$\mathcal{G} = (\gamma^2 \lambda_{qm}^2 \kappa^2) \rho \quad (36)$$

$$(\ln p_i)_t = \left(\frac{V}{\gamma \lambda_{qm}^2 p_i} \right) \frac{\partial \mathcal{G}}{\partial c_i} \quad (37)$$

$$(\ln c_i)_t = \left(\frac{V}{\gamma \lambda_{qm}^2 p_i c_i} \right) \left(\gamma^2 \lambda_{qm}^2 \kappa^2 R^i - p_i \frac{\partial \mathcal{G}}{\partial p_i} \right) \quad (38)$$

where

$$R^i = - \sum_j G^{ij} (\rho + \hat{p}_j) = r^i - \frac{2 \rho}{d-1} . \quad (39)$$

Calculating ρ_t using the above equations for ρ , $\frac{\partial \mathcal{G}}{\partial c_i}$ and $\frac{\partial \mathcal{G}}{\partial p_i}$, and writing $p_i = e^{l_i}$, gives, as it must, the conservation equation (6) :

$$\rho_t = \sum_i R^i (l_i)_t = - \sum_i (\rho + \hat{p}_i) \lambda_t^i . \quad (40)$$

We now specialise to the model where ψ^i s are given by equation (30). It is useful to define the following quantities :

$$S_i = \frac{\partial \mathcal{G}}{\partial \psi^i} = \sum_j G_{ij} \psi^j , \quad g_i = \frac{d f(m^i)}{d m^i} , \quad X_i = g_i S_i \quad (41)$$

and

$$N^{ij} = \frac{d \ln \phi^i}{d \ln p_j} , \quad M^{ij} = \frac{d \ln \bar{\mu}^i}{d \ln p_j} . \quad (42)$$

From the definitions of ψ^i and \mathcal{G} and from equation (31), it follows that

$$\begin{aligned} \sum_i S_i \psi^i &= 2 \mathcal{G} , \quad N^{ij} + M^{ij} + G^{ij} = 0 \\ \frac{d \psi^i}{d c_j} &= g_i \left(\frac{\lambda_{qm} p_i}{V} \right) \delta^{ij} , \quad \frac{d \ln \psi^i}{d \ln p_j} = N^{ij} + \frac{g_i m^i}{f^i} M^{ij} \\ \frac{\partial \mathcal{G}}{\partial c_i} &= \left(\frac{\lambda_{qm} p_i}{V} \right) X_i , \quad p_i \frac{\partial \mathcal{G}}{\partial p_i} = \sum_j S_j \psi^j \left(N^{ji} + \frac{g_j m^j}{f^j} M^{ji} \right) . \end{aligned} \quad (43)$$

Hence, equation (37) and (38) become

$$(\gamma \lambda_{qm}) (\ln p_i)_t = X_i \longrightarrow \lambda_t^i = \frac{\sum_j G^{ij} X_j}{\gamma \lambda_{qm}} \quad (44)$$

$$(\gamma \lambda_{qm}) (\ln c_i)_t = \frac{1}{\phi^i m^i} \left(\gamma^2 \lambda_{qm}^2 \kappa^2 R^i - p_i \frac{\partial \mathcal{G}}{\partial p_i} \right) , \quad (45)$$

from which it follows that $\Lambda_t = \frac{\sum_j X_j}{(d-1) \gamma \lambda_{qm}}$ and that $(m^i)_t$ is given by

$$\begin{aligned} (\gamma \lambda_{qm}) \phi^i (m^i)_t &= \gamma^2 \lambda_{qm}^2 \kappa^2 R^i - \sum_j S_j \psi^j N^{ji} \\ &+ \sum_j \left(\phi^i m^i M^{ij} - \phi^j m^j M^{ji} \right) X_j . \end{aligned} \quad (46)$$

Thus, equations (36), (44), and (45) or (46) are the independent equations of motion for our model.¹⁰ They give the time evolution of all quantities

¹⁰Using the chain rule for differentiation, it is straightforward to obtain the general expressions for ψ_t^i and $(X_i)_t$, and hence for λ_t^i , also. However, they are not illuminating nor are they useful for our purposes here and, hence, will not be presented.

for any given initial values, once the equations of state are known for the pressures \hat{p}_i . Thus, for example, given the values of (p_i, c_i) at some initial time t_{init} , equation (36) gives ρ , equations of state then give \hat{p}_i , equations (44) and (45) give $(p_i)_t$ and $(c_i)_t$ which, in turn, give (p_i, c_i) at time $t_{init} \pm \delta t$. Repeating these steps gives (p_i, c_i) for all time t . The λ^i 's follow from equation (23).

Note that if it were possible to invert equation (44) and obtain m^i in terms of λ^i , then equations (36), (44), and (46) would resemble more closely the standard FRW equations. This inversion is generically not possible even for isotropic case and, hence, the simplest way to understand the evolution equations is as explained in the previous paragraph.

We will now consider several illustrative cases by making specific choices for functions ϕ^i , $\bar{\mu}^i$, and $f^i = f(m^i)$.

$\mathbf{f(m^i) = \alpha m^i}$ and $(d+1)$ dimensional Einstein's equations

Consider first the case where $f^i = f(m^i) = \alpha m^i$ and α is a constant. Then, for any choice of ϕ^i and $\bar{\mu}^i$ obeying the condition (31), it follows that ψ^i , $i = 1, 2, \dots, d$, are given by

$$\psi^i = \alpha \frac{\lambda_{qm} p_i c_i}{V} \longrightarrow H_{grav} = - \frac{\alpha^2}{\gamma^2 \kappa^2 V} \sum_{ij (i < j)} (p_i c_i) (p_j c_j) . \quad (47)$$

Also, $g_i = \alpha$ for all i . Hence, equation (44) gives

$$(\gamma \lambda_{qm}) (\ln p_i)_t = \alpha S_i \longrightarrow \lambda_t^i = \frac{\alpha \psi^i}{\gamma \lambda_{qm}} , \quad (a_i)_t = \frac{\alpha^2 c_i}{\gamma L_i} . \quad (48)$$

This shows that c_i is related to $(a_i)_t$. From equation (45), one obtains

$$(\gamma \lambda_{qm}) (\ln c_i)_t = \frac{\alpha}{\psi^i} \left(\gamma^2 \lambda_{qm}^2 \kappa^2 r^i - S_i \psi^i \right) \quad (49)$$

where, since $g_j m^j = f^j$ now, we have used

$$p_i \frac{\partial \mathcal{G}}{\partial p_i} = - \sum_j S_j \psi^j G^{ij} = - \frac{2 \mathcal{G}}{d-1} + S_i \psi^i .$$

One can now calculate $(\ln \psi^i)_t$ and, after a little algebra, it follows that

$$\psi_t^i + \Lambda_t \psi^i = \alpha (\gamma \lambda_{qm} \kappa^2) r^i . \quad (50)$$

Substituting $\alpha \psi^i = (\gamma \lambda_{qm}) \lambda_t^i$, equations (36) and (50) give

$$\sum_{i,j} G_{ij} \lambda_t^i \lambda_t^j = 2\alpha^2 \kappa^2 \rho \quad (51)$$

$$\lambda_{tt}^i + \Lambda_t \lambda_t^i = \alpha^2 \kappa^2 r^i . \quad (52)$$

For $\alpha = 1$, these are indeed the Einstein's equations (4) and (5) for a $(d+1)$ dimensional homogeneous anisotropic universe. These equations may also be thought of as Einstein's equations with t replaced by αt . Thus, for example, $\alpha = -1$ can be thought of as reversing the direction of time.

$$\mathbf{f}(\mathbf{m}^i) = \alpha (\mathbf{m}^i - \mathbf{m}_{\text{shift}})$$

Consider now the case where the function

$$f^i = f(m^i) = \alpha \tilde{m}^i , \quad \tilde{m}^i = m^i - m_{\text{shift}}$$

and α and m_{shift} are constants, same for all i . The equations (36) and (44) will remain the same when expressed in terms of p_i and \tilde{m}^i . However, equation (46) for $(\tilde{m}^i)_t$ will, in general, be different and will have explicit dependence on the shift constant m_{shift} . In the two examples of ϕ^i to be given below, this dependence will drop out. In these examples then, but not in general, $f(m^i) = \alpha \tilde{m}^i$ will again lead to the Einstein's equations for a $(d+1)$ dimensional homogeneous anisotropic universe.

LQC-inspired equations for $(d+1)$ dimensional cosmology

Although introducing the function ϕ^i renders our model in equation (30) more general, it does not seem to be of much help in obtaining analytical

solutions. Therefore, we will consider only two explicit examples of ϕ^i : One, as in [5, 13, 14],

$$(1) \quad \phi^i = 1 \quad , \quad \bar{\mu}^i = \frac{\lambda_{qm} p_i}{V} \quad \longrightarrow \quad N^{ij} = 0 \quad (53)$$

and another, as in [23],

$$(2) \quad \phi^i = \frac{\lambda_{qm} p_i}{\epsilon V} \quad , \quad \bar{\mu}^i = \epsilon \quad \longrightarrow \quad M^{ij} = 0 \quad (54)$$

where ϵ is a constant, referred to as the discreteness parameter. Note that in these two examples, the equations remain the same under a shift of m^i where $m^i \rightarrow \tilde{m}^i = m^i - m_{shift}$ and m_{shift} is a constant, same for all i . This is because in example (1), $\phi^i = 1$ and $M^{ij} = M^{ji} = -G^{ij}$, and the right hand side of equation (46) depends only on the difference $(m^i - m^j) = (\tilde{m}^i - \tilde{m}^j)$. In example (2), $M^{ij} = 0$ and the right hand side of equation (46) has no explicit dependence on m^i 's. We now consider these two examples in more detail.

Example (1) : $\phi^i = 1 \quad , \quad N^{ij} = 0$

In this case where $\phi^i = 1$, we have $\psi^i = f^i$ and $\bar{\mu}^i = \frac{\lambda_{qm} p_i}{V}$. This expression for $\bar{\mu}^i$ is a $(d + 1)$ dimensional generalisation of that given in [5, 13, 14]. Using $\phi^i = 1$, $M^{ij} = -G^{ij}$, and

$$\sum_j G^{ij} (m^i - m^j) X_j = \frac{\sum_j (m^i - m^j) X_j}{d - 1}$$

in equation (46), it follows that

$$(m^i)_t + \frac{\sum_j (m^i - m^j) X_j}{(d - 1) \gamma \lambda_{qm}} = \gamma \lambda_{qm} \kappa^2 R^i \quad . \quad (55)$$

Thus, equations (36), (44), and (55) may be taken to be the equations of motion in the case where $\phi^i = 1$.

Note that setting $f^i = \sin m^i$, $i = 1, 2, \dots, d$, gives the analog of the effective LQC equations, generalised now to a $(d+1)$ dimensional homogeneous anisotropic universe. The corresponding H_{grav} is given by ¹¹

$$H_{grav} = - \frac{V}{\gamma^2 \lambda_{qm}^2 \kappa^2} \sum_{ij (i < j)} (\sin m^i) (\sin m^j) .$$

Also note that, when $d = 3$ and $f^i = \sin m^i$, the expression for $(c_i)_t$ agrees with that given in [13] for $\hat{p}_i = \rho$ which is the equation of state for a massless scalar field considered there; and, that the expression for $(m^i)_t$ agrees with that given in [14] for isotropic pressures, namely for $\hat{p}_i = \hat{p}$.

Example (2) : $\bar{\mu}^i = \epsilon$, $\mathbf{M}^{\mathbf{j}} = \mathbf{0}$

In this case we have $\phi^i = \frac{\lambda_{qm} p_i}{\epsilon V}$ and $\psi^i = \phi^i f^i$ where ϵ is a constant, referred to as the discreteness parameter in [23]. Using $N^{ij} = -G^{ij}$,

$$\sum_j G^{ij} (S_j \psi^j) = \frac{2 \mathcal{G}}{d-1} - S_i \psi^i ,$$

and $R^i + \frac{2\rho}{d-1} = r^i$ in equation (46), it follows that

$$\gamma \lambda_{qm} \phi^i (m^i)_t = (\gamma^2 \lambda_{qm}^2 \kappa^2) r^i - S_i \psi^i \quad (56)$$

and, after a little algebra, further that

$$\psi_t^i + \Lambda_t \psi^i = (\gamma \lambda_{qm} \kappa^2) g_i r^i . \quad (57)$$

Thus, equations (36), (44), and (57) may be taken to be the equations of motion in the case where $\bar{\mu}^i = \epsilon$.

Note that, when $d = 3$ and $f^i = \sin m^i$, these expressions agree with those given in [23] for $\hat{p}_i = \rho = 0$.

¹¹Our preliminary analysis [11] suggests that these $(d+1)$ dimensional LQC analogs with $f^i = \sin m^i$ can be derived from an underlying theory, namely from the $(d+1)$ dimensional LQG formulation given in [8, 9, 10]. Isotropic case arising from this formulation has been considered in [25].

Isotropic case

Consider the isotropic case where

$$\hat{p}^i = \hat{p} \ , \quad (p^i, c_i) = (p, c) \ , \quad a_i = a \ .$$

Then for any choice of $\phi^i = \phi$ and $\bar{\mu}^i = \bar{\mu}$, we have, for example,

$$(m^i; \psi^i; g_i, S_i, X_i) = (m; \psi; g, S, X) \ , \quad \lambda_t^i = \frac{a_t}{a}$$

and, hence,

$$2 \mathcal{G} = d(d-1) \psi^2 \ , \quad S = (d-1) \psi \ , \quad X = g S \ .$$

Equations (36) and (44) then give

$$\psi^2 = \frac{2 \gamma^2 \lambda_{qm}^2 \kappa^2}{d(d-1)} \rho = \frac{\rho}{\rho_{ub}} \ , \quad \frac{a_t}{a} = \frac{g \psi}{\gamma \lambda_{qm}} \quad (58)$$

where $\rho_{ub} = \frac{d(d-1)}{2 \gamma^2 \lambda_{qm}^2 \kappa^2}$. One now obtains

$$\left(\frac{a_t}{a} \right)^2 = \frac{2 \kappa^2}{d(d-1)} (\rho g^2) \ . \quad (59)$$

Equation (45) or (46) gives an expression for c_t or m_t . To proceed further, ϕ and $\bar{\mu}$ are needed. In example (1) where $\phi = 1$, it follows from equation (55) that

$$m_t = - \frac{\gamma \lambda_{qm} \kappa^2}{d-1} (\rho + \hat{p}) \quad (60)$$

where we have used $m^i = m$ and $R^i = R = - \frac{\rho + \hat{p}}{d-1}$. Also, note that when $\phi = 1$ and $\psi(m) = f(m) = \sin m$, we have $g^2 = \cos^2 m = 1 - \frac{\rho}{\rho_{ub}}$. Equation (59) then gives the $(d+1)$ dimensional result obtained in [25].

4. A few explicit solutions

Equations of motion (36), (44), and (45) can be solved numerically for any choice of the functions ϕ^i and $f(x)$, as explained below equation (46). The general features of these equations can also be seen easily. For example: **(a)** In the limit $f(x) \rightarrow x$, the equations of motion become same as Einstein's equations. **(b)** The density ρ is bounded from above if the functions ψ^i are. That is,

$$\psi^i \leq \psi_{mx} \implies \rho \leq \rho_{ub}$$

where $\rho_{ub} = \frac{d(d-1)}{2\gamma^2\lambda_{qm}^2\kappa^2} (\psi_{mx}^2)$. In example **(1)**, $\psi^i = f^i = \sin m^i$ and, hence, $\psi_{mx} = 1$. In a given anisotropic evolution ρ may not reach ρ_{ub} since not all ψ^i may reach the maximum value ψ_{mx} at the same time.

In general, it is not possible to obtain explicit analytical solutions to the equations of motion. They can, however, be obtained in a few special cases. We will now consider such cases.

Anisotropic case in Example (2) with stiff matter

Consider the anisotropic case in example **(2)** where $\bar{\mu}^i = \epsilon$. Let the equations of state be given by that of a stiff matter, namely by $\hat{p}^i = \rho$.¹² Then $r^i = 0$ and equation (57) can be integrated to obtain

$$\psi^i = \frac{\lambda_{qm}}{\epsilon V} K^i \iff f^i = f(m^i) = \frac{K^i}{p_i} . \quad (61)$$

where K^i are integration constants. With no loss of generality, we assume that K^i are all strictly positive, namely that $K^i > 0$ for all i . Note then that p_i s are bounded from below if the functions f^i are bounded from above. That is,

$$f^i \leq f_{mx} \implies p_i \geq \frac{K^i}{f_{mx}} .$$

¹²There is not much loss of generality in considering stiff matter equations of state. This is because the modifications in the effective equations are expected to become important only when the volume is small and the densities are high. In this limit, the dominant matter fields will be those with the highest $w = \frac{\hat{p}^i}{\rho}$, namely $w = 1$.

Also, we have

$$S_i = \frac{\lambda_{qm}}{\epsilon V} \sigma^i , \quad \sigma_i = \sum_j G_{ij} K^j , \quad X_i = g_i S_i . \quad (62)$$

Equation (44) then gives

$$\frac{dp_i}{g_i p_i} = \frac{\sigma^i}{\gamma \epsilon} d\tau , \quad dt = V d\tau . \quad (63)$$

Thus, for a given function $f(x)$, inverting equation (61) gives m^i and then $g_i = \frac{df(m^i)}{dm^i}$ in terms of K^i and p_i . Then integrating equation (63) gives p_i in terms of τ . Using $V = (\prod_i p_i)^{\frac{1}{d-1}}$ in equation (63) gives t in terms of τ . Then equation (22) gives the scale factors a_i and equation (36) gives ρ :

$$a_i = \frac{V}{p_i L_i} , \quad \rho = \left(\frac{\sum_{ij} G_{ij} K^i K^j}{2 \gamma^2 \epsilon^2 \kappa^2} \right) \frac{1}{V^2} . \quad (64)$$

It follows from the above expressions that if the p_i s are bounded from below by $p_{min} > 0$, that is if $p_i \geq p_{min} > 0$, then the volume V will not vanish and the density ρ will remain finite and not diverge.

$$(i) \quad \mathbf{f}(\mathbf{m}^i) = \sin m^i$$

Consider the case where $f^i = f(m^i) = \sin m^i$. In this case, we have

$$g_i = \cos m^i , \quad g_i p_i = \sqrt{p_i^2 - (K^i)^2} .$$

Equation (63) for p_i can be easily integrated and one obtains

$$p_i = K^i \cosh \theta_i , \quad \sin m^i = \frac{1}{\cosh \theta_i} .$$

where $\theta_i(\tau)$ are given by

$$\theta_i = \frac{\sigma^i}{\gamma \epsilon} (\tau - \tau_0) + \theta_{i0} = \frac{\sigma^i}{\gamma \epsilon} (\tau - \tau_i) .$$

In the above equation, we have assumed that $p_i = p_{i0} = K^i \cosh \theta_{i0}$ at an initial time τ_0 , and the second equality defines τ_i . Let $t = t_0$ be the initial time at τ_0 . Then equation (63) for t gives

$$t - t_0 = \int_{\tau_0}^{\tau} d\tau \left(\prod_i p_i \right)^{\frac{1}{d-1}} .$$

Note that $p_i(\tau) \geq p_i(\tau_i) = K^i > 0$. Hence it follows that the volume V will not vanish and, from equation (64), that the density ρ will not diverge. It is straightforward to see that the variable τ can range between $-\infty$ and $+\infty$; the θ_i s and t range between $-\infty$ and $+\infty$; and the m^i s between 0 and π . When all the m^i s are near 0 or π , we have $g_i = +1$ or -1 for all i . The evolution is then same as that given by Einstein's equations. The precise details of the evolution depend on the initial values K^i .

$$(ii) \quad f(\mathbf{m}^i) = \frac{\alpha \mathbf{m}^i}{m_*} \left(2 - \frac{m^i}{m_*} \right)$$

A closer inspection of the results in the previous case shows that the salient features of the evolution there are due to the fact that the function $f(x) = \sin x$ starts linearly near $x = 0$; reaches a maximum; and then decreases and reaches a zero again linearly at $x = \pi$. This suggests that any function with these properties must also result in similar salient features of the evolution. In order to illustrate this explicitly, consider a case with these properties where now $f^i = f(m^i) = \frac{\alpha m^i}{m_*} \left(2 - \frac{m^i}{m_*} \right)$ and α and m_* are positive constants, same for all i . In this case, we have

$$g_i = \frac{2\alpha}{m_*} \left(1 - \frac{m^i}{m_*} \right) = \frac{2\alpha}{m_*} \epsilon_i \sqrt{1 - \frac{f^i}{\alpha}}$$

where $\epsilon_i = \text{sgn} \left(1 - \frac{m^i}{m_*} \right)$ and

$$g_i p_i = \frac{2\alpha}{m_*} \epsilon_i \sqrt{p_i^2 - \frac{K^i p_i}{\alpha}} .$$

Equation (63) for p_i can be easily integrated and, after a little algebra, one obtains

$$p_i = \frac{K^i}{\alpha} \cosh^2 \theta_i$$

where $\theta_i(\tau)$ are given by

$$\theta_i = \frac{\alpha}{m_*} \frac{\sigma^i}{\gamma \epsilon} (\tau - \tau_0) + \theta_{i0} = \frac{\alpha}{m_*} \frac{\sigma^i}{\gamma \epsilon} (\tau - \tau_i) .$$

In the above equation, we have assumed that $p_i = p_{i0} = \frac{K^i}{\alpha} \cosh^2 \theta_{i0}$ at an initial time τ_0 , and the second equality defines τ_i . Let $t = t_0$ be the initial time at τ_0 . Then equation (63) gives

$$t - t_0 = \int_{\tau_0}^{\tau} d\tau \left(\prod_i p_i \right)^{\frac{1}{d-1}} .$$

Note that $p_i(\tau) \geq p_i(\tau_i) = \frac{K^i}{\alpha} > 0$. Hence it follows that the volume V will not vanish and, from equation (64), that the density ρ will not diverge. It is straightforward to see that the variable τ can range between $-\infty$ and $+\infty$; the θ_i s and t range between $-\infty$ and $+\infty$; and the m^i s between 0 and $2m_*$. When all the m^i s are near 0 or $2m_*$, we have $g_i = +\frac{2\alpha}{m_*}$ or $-\frac{2\alpha}{m_*}$ for all i . The evolution is then same as that given by Einstein's equations. The precise details of the evolution depend on the initial values K^i .

$$(iii) \quad f(\mathbf{m}^i) = \alpha \left(1 - \left(1 - \frac{\mathbf{m}^i}{m_*} \right)^{2n} \right)$$

We now consider another case where $f(m^i)$ starts from zero, reaches a maximum, and falls back to zero, but now its shape near the maximum can be made flatter. Hence, consider a case where now $f^i = f(m^i) = \alpha \left(1 - \left(1 - \frac{m^i}{m_*} \right)^{2n} \right)$, α and m_* are positive constants, same for all i , and n is a positive integer. The previous example corresponds to $n = 1$. The maximum, which is at m_* , will be flatter for larger values of n . In this case, we have

$$g_i = \frac{2n\alpha}{m_*} \left(1 - \frac{m^i}{m_*} \right)^{2n-1} = \frac{2n\alpha}{m_*} \epsilon_i \left(1 - \frac{f^i}{\alpha} \right)^{\frac{2n-1}{2n}}$$

where $\epsilon_i = \text{sgn} \left(1 - \frac{m^i}{m_*} \right)$ and

$$g_i p_i = \frac{2n\alpha}{m_*} \epsilon_i p_i^{\frac{1}{2n}} \left(p_i - \frac{K^i}{\alpha} \right)^{\frac{2n-1}{2n}} .$$

We are not able to integrate equation (63) for p_i now for arbitrary values of integer n . However, one can obtain straightforwardly the leading behaviour near the zeros and the maximum of f^i , namely near $m^i = 0$, m_* , and $2m_*$. Near $m^i = 0$, the function f is of the type $f(m^i) = \alpha m^i$ and, hence, the earlier analysis carries over. Near $m^i = 2m_*$, the function f is of the type $f(m^i) = \alpha(m^i - m_{shift})$. Note that for both the anisotropic examples (1) and (2), the form of the equations of motion, see equations (55) and (56) for $(m^i)_t$ in particular, remain the same under a constant shift in m^i with the shift being the same for all i . It is then clear that, after a constant shift in m^i , the earlier analysis carries over near $m^i = 2m_*$ also.

Consider when m^i , for a given i , is near m_* and τ is near τ_i where τ_i is defined by $m^i(\tau_i) = m_*$. The corresponding $f^i \lesssim \alpha$ and, in the limit $\tau \simeq \tau_i$, let

$$p_i = \frac{K^i}{\alpha} (1 + x^i) \quad , \quad x^i \ll 1 \quad .$$

One can then integrate equation (63) in this limit. After a straightforward algebra, it follows that

$$x^i = \left(\frac{\alpha \sigma^i}{m_* \gamma \epsilon} \right)^{2n} (\tau - \tau_i)^{2n} \quad .$$

Note that $p_i(\tau) \geq p_i(\tau_i) = \frac{K^i}{\alpha} > 0$. Hence it follows that the volume V will not vanish and, from equation (64), that the density ρ will not diverge. It can be seen by a straightforward but qualitative analysis that the variable τ can range between $-\infty$ and $+\infty$, t ranges between $-\infty$ and $+\infty$, and the m^i 's between 0 and $2m_*$. When all the m^i 's are near 0 or $2m_*$, we have $g_i = +\frac{2n\alpha}{m_*}$ or $-\frac{2n\alpha}{m_*}$ for all i . The evolution is then same as that given by Einstein's equations. The precise details of the evolution depend on the value of n and the initial values K^i .

$$(iv) \quad \mathbf{f}(\mathbf{m}^i) = \alpha (1 - \mathbf{e}^{-\beta \mathbf{m}^i})$$

We now consider another case where $f(m^i)$ starts from zero and increases monotonically to a constant value. Hence, consider a case where now $f^i = f(m^i) = \alpha (1 - e^{-\beta m^i})$ and α and β are positive constants, same for all i . In this case, we have

$$g_i = \alpha \beta e^{-\beta m^i} = \beta (\alpha - f^i) \quad , \quad g_i p_i = \beta (\alpha p_i - K^i) \quad .$$

Equation (63) for p_i can be easily integrated and, incorporating the condition that $p_i = p_{i0}$ at an initial time τ_0 , one obtains

$$\ln (\alpha p_i - K^i) = \frac{\alpha \beta \sigma^i}{\gamma \epsilon} (\tau - \tau_0) + \ln (\alpha p_{i0} - K^i) .$$

Let $t = t_0$ be the initial time at τ_0 . Then equation (63) gives

$$t - t_0 = \int_{\tau_0}^{\tau} d\tau \left(\prod_i p_i \right)^{\frac{1}{d-1}} .$$

Note that $p_i(\tau) = \frac{K^i}{f^i} \geq p_i(-\infty) = \frac{K^i}{\alpha} > 0$. Hence it follows that the volume V will not vanish and, from equation (64), that the density ρ will not diverge. Also, it is straightforward to see that as p_i ranges between $+\infty$ and $\frac{K^i}{\alpha}$, the variables τ and t range between $+\infty$ and $-\infty$, and the m^i 's range between 0 and $+\infty$. When all the m^i 's are near 0 or $+\infty$, we have $g_i = \alpha\beta$ or 0 for all i . The evolution is then same as that given by Einstein's equations. The precise details of the evolution depend on the initial values K^i . However, note that, as $m^i \rightarrow \infty$, $f(m^i)$ in this case approaches the maximum asymptotically and does not decrease from its maximum. As $\tau \rightarrow -\infty$, it is easy to see that all the m^i 's approach ∞ and g_i 's all vanish and, hence, that the density ρ and the scale factors a^i reach their finite, non zero constant values asymptotically. Consequently, there is no bounce where the scale factors increase again from their minimum. This phase of the evolution, where the density and volume remain constant, is similar to the 'Hagedorn phase' in string/M theory where, as one goes back in time, the universe's temperature $\rightarrow \mathcal{O}(l_s^{-1})$ and its density $\rightarrow \mathcal{O}(l_s^{-(d+1)})$ asymptotically, l_s being the string length scale [15] – [22], [12].

Isotropic case in Example (1)

Consider the isotropic case in example (1) where $\phi = 1$ and $\psi = f(m)$. Let the equation of state be given by $\hat{p} = w \rho$ where w is a constant. Then equations (58) and (60) give

$$-\frac{dm}{f^2} = \frac{d(1+w)}{2\gamma\lambda_{qm}} dt , \quad \frac{da}{a} = \frac{g f}{\gamma\lambda_{qm}} dt . \quad (65)$$

Thus, for a given function $f(m)$, integrating equation (65) gives m in terms of t . Defining $\mathcal{F}(m)$ and incorporating the conditions that $m = m_0$ and $\mathcal{F} = \mathcal{F}_0$ at an initial time $t = t_0$, we write

$$\mathcal{F} = - \int \frac{dm}{f^2} = \frac{d(1+w)}{2\gamma\lambda_{qm}} (t - t_0) + \mathcal{F}_0 . \quad (66)$$

Sometimes, \mathcal{F} may also be written as

$$\mathcal{F} = \frac{d(1+w)}{2\gamma\lambda_{qm}} (t - t_{ub})$$

where $t = t_{ub}$ is the time when $\mathcal{F} = 0$. One can now obtain $f(m)$ and $g(m) = \frac{df}{dm}$ in terms of t . Then another integration will give the scale factor $a(t)$. Or, alternatively, equations (40) and (58) give the scale factor a directly in terms of f :

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-d(1+w)} = \rho_{ub} f^2 \quad (67)$$

where $\rho_{ub} = \frac{d(d-1)}{2\gamma^2\lambda_{qm}^2\kappa^2}$ and we have incorporated the condition that $\rho = \rho_0$ and $a = a_0$ at an initial time $t = t_0$. Thus, for a given $f(m)$, finding \mathcal{F} and expressing f in terms of \mathcal{F} , where possible, constitute an explicit solution.

(i) $\mathbf{f(m) = sin m}$

In the case where $f(m) = \sin m$, one obtains that

$$\mathcal{F} = \cot m \quad , \quad f^2 = \frac{1}{1 + \mathcal{F}^2} .$$

Note that $\mathcal{F}(t)$ and $f(t)$ are of the form

$$\mathcal{F} = \frac{d(1+w)}{2\gamma\lambda_{qm}} (t - t_{ub}) \quad , \quad f^2 = \frac{1}{1 + c_1(t - t_{ub})^2}$$

where $t = t_{ub}$ is the time when $m = \frac{\pi}{2}$ and $\mathcal{F} = 0$. The scale factor $a(t)$ is given by equation (67).

It is clear that $f \leq 1$ and the scale factor a , and hence the volume V , will not vanish. Also, the density ρ will not diverge. It is straightforward to see that as t ranges between $+\infty$ and $-\infty$, m ranges between 0 and π . When m is near 0 or π , it is easy to see that the evolution is same as that given by Einstein's equations.

$$(ii) \quad f(m) = \frac{\alpha m}{m_*} \left(2 - \frac{m}{m_*} \right)$$

In the case where $f(m) = \alpha z (2 - z)$ and $z = \frac{m}{m_*}$, one obtains that

$$\mathcal{F} = \frac{m_*}{4\alpha^2} \left(\frac{2(1-z)}{z(2-z)} + \ln \frac{2-z}{z} \right) = \frac{d(1+w)}{2\gamma\lambda_{qm}} (t - t_0) + \mathcal{F}_0,$$

which can also be written as

$$\mathcal{F} = \frac{m_*}{4\alpha^2} \left(\frac{2(1-z)}{z(2-z)} + \ln \frac{2-z}{z} \right) = \frac{d(1+w)}{2\gamma\lambda_{qm}} (t - t_{ub})$$

where $t = t_{ub}$ is the time when $m = m_*$, $z = 1$, and $\mathcal{F} = 0$. The scale factor $a(t)$ is given by equation (67).

We can not express $f(m)$ analytically in terms of \mathcal{F} in this case. The features of the evolution can, however, be read off easily. As $m \rightarrow 0$, we have $z \rightarrow 0$ and $t \rightarrow +\infty$. As $m \rightarrow m_*$, we have $z \rightarrow 1$ and $t \rightarrow t_{ub}$. As $m \rightarrow 2m_*$, we have $z \rightarrow 2$ and $t \rightarrow -\infty$. Thus, as t decreases from $+\infty$ to t_{ub} to $-\infty$, we have that $f(m)$ increases from 0 to α and then decreases to 0 and, hence from equation (67), that the scale factor $a(t)$ decreases from $+\infty$ to a non zero minimum and then increases to $+\infty$. The volume V will not vanish and the density ρ will not diverge. When m is near 0 or $2m_*$, it is easy to see that the evolution is same as that given by Einstein's equations.

$$(iii) \quad f(m) = \alpha \left(1 - \left(1 - \frac{m}{m_*} \right)^{2n} \right)$$

In the case where $f(m) = \alpha(1 - (1 - z)^{2n})$, $z = \frac{m}{m_*}$, and n is a positive integer, we can not obtain \mathcal{F} explicitly for arbitrary values of n . The features

of the evolution are, however, similar to those in the previous case and can be read off easily. As $m \rightarrow 0$, we have $z \rightarrow 0$, $f \sim z$, and

$$\mathcal{F} \sim - \int \frac{dz}{z^2} \sim t \rightarrow +\infty .$$

As $m \rightarrow 2m_*$ from below, we have $z \rightarrow 2$ from below, $f \sim (2 - z)$, and

$$\mathcal{F} \sim - \int \frac{dz}{(2 - z)^2} \sim t \rightarrow -\infty .$$

As $m \rightarrow m_*$, we have $z = 1 - x \rightarrow 1$ and $t \rightarrow t_{ub}$. Then $f = \alpha(1 - x^{2n})$ and

$$\mathcal{F} = \frac{m_*}{4\alpha^2} (x + \dots) = \frac{d(1 + w)}{2\gamma\lambda_{qm}} (t - t_{ub}) .$$

Thus, as t decreases from $+\infty$ to t_{ub} to $-\infty$, we have that $f(m)$ increases from 0 to α and then decreases to 0 and, hence from equation (67), that the scale factor $a(t)$ decreases from $+\infty$ to a non zero minimum and then increases to $+\infty$. The volume V will not vanish and the density ρ will not diverge. When m is near 0 or $2m_*$, it is easy to see that the evolution is same as that given by Einstein's equations.

$$(iv) \quad \mathbf{f}(\mathbf{m}) = \alpha (1 - \mathbf{e}^{-\beta \mathbf{m}})$$

Consider the case where $f(m) = \alpha (1 - e^{-\beta m})$ and α and β are positive constants. For $m \geq 0$, the function starts linearly and increases monotonically to a constant value α as $m \rightarrow \infty$. Unlike other cases, in this case the function $f(m)$ approaches the maximum asymptotically and does not decrease. It is straightforward to perform the integration $\int \frac{dm}{f^2}$ and one obtains that,

$$\mathcal{F} = \frac{1}{\alpha^2 \beta} \left(\frac{1}{1 - e^{-\beta m}} - \ln(e^{\beta m} - 1) \right) = \frac{d(1 + w)}{2\gamma\lambda_{qm}} (t - t_0) + \mathcal{F}_0 .$$

The scale factor $a(t)$ is given by equation (67).

We can not express $f(m)$ analytically in terms of \mathcal{F} in this case. The features of the evolution can, however, be read off easily. As $m \rightarrow 0$, we have

$\mathcal{F} \rightarrow \left(\frac{1}{\alpha^2 \beta^2}\right) \frac{1}{m}$ and $t \rightarrow +\infty$. As m increases, \mathcal{F} decreases monotonically and, as $m \rightarrow +\infty$, we have $\mathcal{F} \rightarrow -\frac{m}{\alpha^2}$ and $t \rightarrow -\infty$. Thus, as t decreases from $+\infty$ to $-\infty$, we have that $f(m)$ increases monotonically from 0 to α and, hence from equation (67), that the scale factor $a(t)$ decreases monotonically from $+\infty$ to a non zero minimum. The volume V will not vanish and the density ρ will not diverge. When m is near 0, it is easy to see that the evolution is same as that given by Einstein's equations. Note that, as $m \rightarrow \infty$, $f(m)$ in this case approaches the maximum asymptotically and does not decrease from its maximum. It is easy to see that the density ρ and the scale factor a reach their finite, non zero constant values asymptotically. Consequently, there is no bounce where the scale factor increases again from its minimum. This phase of the evolution, where the density and volume remain constant, is similar to the 'Hagedorn phase' in string/M theory where, as one goes back in time, the universe's temperature $\rightarrow \mathcal{O}(l_s^{-1})$ and its density $\rightarrow \mathcal{O}(l_s^{-(d+1)})$ asymptotically, l_s being the string length scale [15] – [22], [12].

5. Conclusion

We now summarise the present paper. We consider a $(d+1)$ dimensional homogeneous anisotropic universe. In Einstein's theory, it has generically a big-bang singularity in the past. In $(3+1)$ dimensions, its dynamics is modified by LQC and then it has generically a big bounce in the past, instead of a big-bang singularity. This dynamics, modified by the quantum effects, can be well described by effective equations of motion.

In this paper, we generalise these effective equations to $(d+1)$ dimensions. They may then describe the modified dynamics of a $(d+1)$ dimensional homogeneous anisotropic universe. The generalisation is natural and straightforward but empirical, and involves two functions. These generalised equations may be considered as a class of LQC – inspired models for $(d+1)$ dimensional early universe cosmology.

The matter Hamiltonian, in both LQC and in the models presented here, may include various types of scalar fields and other matter fields. But it is assumed to depend only on p_i and not on c_i . Since c_i is related to $(a_i)_t$, this means that matter fields couple to the metric fields but not to the

curvatures. This assumption also leads to the standard conservation equation (6) irrespective of what H_{grav} .

Special cases of the functions in the present models lead to Einstein's equations in $(d+1)$ dimensions and to the effective LQC equations in $(3+1)$ dimensions. One can also obtain a universe which has neither a big bang singularity nor a big bounce but approaches asymptotically, as $t \rightarrow -\infty$, a 'Hagedorn like' phase where its density and volume remain constant. In a few special cases, we also obtain explicit solutions to the equations of motion.

We conclude now by mentioning a few issues for further studies.

(a) In LQC, as well as in our generalisations, matter sector remains 'classical'. Quantum effects in the matter sector should also be included.

(b) In Einstein's theory, an $(n+3+1)$ dimensional universe with n compact spatial directions of sufficiently small sizes may be thought of as effectively a $(3+1)$ dimensional universe but with extra scalar fields appearing in the matter sector which describe the sizes of the compact directions. There must be an analog of this in the present generalised equations which, however, is not clear to us. Understanding this effective lowering of dimensions may provide an insight into the issue (a) mentioned above.

(c) One may also obtain LQG – inspired modifications to Oppenheimer – Volkoff equations which describe the static spherically symmetric stars. One can then study their effects on, for example, the maximum mass of a stable star. Similarly for Oppenheimer – Snyder equations for stellar collapse.

(d) The previous two issues may both be subsumed by finding LQG – inspired modifications to the $(d+1)$ dimensional Einstein's equations (3) in a covariant form. See [26, 27, 28] for a study of such modifications to the $(3+1)$ dimensional Einstein's equations in a covariant form, obtained from higher curvature effective actions. Such modified covariant equations, even if obtained only at an empirical level, can be used in a variety of other contexts also, for example in studying the evolution of the inhomogeneous perturbations in a universe undergoing bounce.

Acknowledgement: We thank G. Date for discussions and helpful suggestions, and Arnab Priya Saha for pointing out the references [8, 9, 10].

References

- [1] M. Bojowald, *Absence of singularity in loop quantum cosmology*,
Phys. Rev. Lett. **86**, 5227 (2001)
doi:10.1103/PhysRevLett.86.5227 [gr-qc/0102069];
The Inverse scale factor in isotropic quantum geometry,
Phys. Rev. D **64**, 084018 (2001)
doi:10.1103/PhysRevD.64.084018 [gr-qc/0105067];
Isotropic loop quantum cosmology,
Class. Quant. Grav. **19**, 2717 (2002)
doi:10.1088/0264-9381/19/10/313 [gr-qc/0202077];
Homogeneous loop quantum cosmology,
Class. Quant. Grav. **20**, 2595 (2003)
doi:10.1088/0264-9381/20/13/310 [gr-qc/0303073].
- [2] A. Ashtekar, M. Bojowald and J. Lewandowski,
Mathematical structure of loop quantum cosmology,
Adv. Theor. Math. Phys. **7**, no. 2, 233 (2003)
doi:10.4310/ATMP.2003.v7.n2.a2 [gr-qc/0304074].
- [3] A. Ashtekar, T. Pawłowski and P. Singh,
Quantum nature of the big bang,
Phys. Rev. Lett. **96**, 141301 (2006)
doi:10.1103/PhysRevLett.96.141301 [gr-qc/0602086];
Quantum Nature of the Big Bang: Improved dynamics,
Phys. Rev. D **74**, 084003 (2006)
doi:10.1103/PhysRevD.74.084003 [gr-qc/0607039].
- [4] A. Ashtekar, *New Variables for Classical and Quantum Gravity*,
Phys. Rev. Lett. **57**, 2244 (1986) doi:10.1103/PhysRevLett.57.2244;
New Hamiltonian Formulation of General Relativity,
Phys. Rev. D **36**, 1587 (1987) doi:10.1103/PhysRevD.36.1587.
- [5] A. Ashtekar and P. Singh, *Loop Quantum Cosmology: A Status Report*,
Class. Quant. Grav. **28**, 213001 (2011)
doi:10.1088/0264-9381/28/21/213001 [arXiv:1108.0893 [gr-qc]].
- [6] M. Varadarajan, *On the resolution of the big bang singularity in isotropic Loop Quantum Cosmology*,

- Class. Quant. Grav. **26**, 085006 (2009)
doi:10.1088/0264-9381/26/8/085006 [arXiv:0812.0272 [gr-qc]].
- [7] G. Date, *Lectures on LQG/LQC*, arXiv:1004.2952 [gr-qc].
- [8] N. Bodendorfer, T. Thiemann and A. Thurn, *New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis*, Class. Quant. Grav. **30**, 045001 (2013)
doi:10.1088/0264-9381/30/4/045001 [arXiv:1105.3703 [gr-qc]].
- [9] N. Bodendorfer, T. Thiemann and A. Thurn, *New Variables for Classical and Quantum Gravity in all Dimensions II. Lagrangian Analysis*, Class. Quant. Grav. **30**, 045002 (2013)
doi:10.1088/0264-9381/30/4/045002 [arXiv:1105.3704 [gr-qc]].
- [10] N. Bodendorfer, T. Thiemann and A. Thurn, *New Variables for Classical and Quantum Gravity in all Dimensions III. Quantum Theory*, Class. Quant. Grav. **30**, 045003 (2013)
doi:10.1088/0264-9381/30/4/045003 [arXiv:1105.3705 [gr-qc]].
- [11] S. Kalyana Rama and Arnab Priya Saha, *Unpublished notes*.
- [12] S. Bhowmick and S. Kalyana Rama,
10 + 1 to 3 + 1 in an Early Universe with mutually BPS Intersecting Branes, Phys. Rev. D **82**, 083526 (2010)
doi:10.1103/PhysRevD.82.083526 [arXiv:1007.0205 [hep-th]].
- [13] A. Ashtekar and E. Wilson-Ewing,
Loop quantum cosmology of Bianchi I models,
Phys. Rev. D **79**, 083535 (2009)
doi:10.1103/PhysRevD.79.083535 [arXiv:0903.3397 [gr-qc]].
- [14] L. Linsefors and A. Barrau, *Modified Friedmann equation and survey of solutions in effective Bianchi-I loop quantum cosmology*, Class. Quant. Grav. **31**, 015018 (2014)
doi:10.1088/0264-9381/31/1/015018 [arXiv:1305.4516 [gr-qc]].
- [15] M. J. Bowick and L. C. R. Wijewardhana,
Superstrings at High Temperature, Phys. Rev. Lett. **54**, 2485 (1985).
doi:10.1103/PhysRevLett.54.2485 ;

- Superstring Gravity and the Early Universe*,
Gen. Rel. Grav. **18**, 59 (1986). doi:10.1007/BF00843749.
- [16] R. H. Brandenberger and C. Vafa, *Superstrings in the Early Universe*,
Nucl. Phys. B **316**, 391 (1989). doi:10.1016/0550-3213(89)90037-0
 - [17] A. A. Tseytlin and C. Vafa, *Elements of string cosmology*,
Nucl. Phys. B **372**, 443 (1992)
doi:10.1016/0550-3213(92)90327-8 [hep-th/9109048]
 - [18] G. Veneziano, *A Model for the big bounce*, JCAP **0403**, 004 (2004)
doi:10.1088/1475-7516/2004/03/004 [hep-th/0312182].
 - [19] A. Nayeri, R. H. Brandenberger and C. Vafa,
Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology, Phys. Rev. Lett. **97**, 021302 (2006)
doi:10.1103/PhysRevLett.97.021302 [hep-th/0511140].
 - [20] S. Kalyana Rama, *A Stringy correspondence principle in cosmology*,
Phys. Lett. B **638**, 100 (2006)
doi:10.1016/j.physletb.2006.05.047 [hep-th/0603216];
A Principle to Determine the Number (3 + 1) of Large Spacetime Dimensions, Phys. Lett. B **645**, 365 (2007)
doi:10.1016/j.physletb.2006.11.077 [hep-th/0610071].
 - [21] B. D. Chowdhury and S. D. Mathur, *Fractional Brane State in the Early Universe*, Class. Quant. Grav. **24**, 2689 (2007)
doi:10.1088/0264-9381/24/10/014 [hep-th/0611330].
 - [22] S. Kalyana Rama, *Entropy of anisotropic universe and fractional branes*, Gen. Rel. Grav. **39**, 1773 (2007)
doi:10.1007/s10714-007-0488-1 [hep-th/0702202 [HEP-TH]];
Consequences of U dualities for Intersecting Branes in the Universe, Phys. Lett. B **656**, 226 (2007)
doi:10.1016/j.physletb.2007.09.069 [arXiv:0707.1421 [hep-th]].
 - [23] G. Date, *Absence of the Kasner singularity in the effective dynamics from loop quantum cosmology*, Phys. Rev. D **71**, 127502 (2005)
doi:10.1103/PhysRevD.71.127502 [gr-qc/0505002].

- [24] N. Bodendorfer, *Black hole entropy from loop quantum gravity in higher dimensions*, Phys. Lett. B **726**, 887 (2013)
doi:10.1016/j.physletb.2013.09.043 [arXiv:1307.5029 [gr-qc]].
- [25] X. Zhang, *Higher dimensional Loop Quantum Cosmology*, Eur. Phys. J. C **76**, no. 7, 395 (2016)
doi:10.1140/epjc/s10052-016-4249-8 [arXiv:1506.05597 [gr-qc]].
- [26] T. P. Sotiriou, *Covariant Effective Action for Loop Quantum Cosmology from Order Reduction*, Phys. Rev. D **79**, 044035 (2009)
doi:10.1103/PhysRevD.79.044035 [arXiv:0811.1799 [gr-qc]].
- [27] G. Date and S. Sengupta, *Effective Actions from Loop Quantum Cosmology: Correspondence with Higher Curvature Gravity*, Class. Quant. Grav. **26**, 105002 (2009)
doi:10.1088/0264-9381/26/10/105002 [arXiv:0811.4023 [gr-qc]].
- [28] R. C. Helling, *Higher curvature counter terms cause the bounce in loop cosmology*, arXiv:0912.3011 [gr-qc].